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Duality and double shuffle relations of multiple zeta values

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Abstract

We prove that certain families of duality relations of the multiple zeta values (MZV's) are consequences of the extended double shuffle relations (EDSR's), thereby proving a part of the conjecture that the EDSR's give all linear relations of the MZV's.

Keywords: Multiple zeta value; Duality; Double shuffle relation

1. Introduction

The multiple zeta values (MZV's) are defined for positive integers $k_1, \dots, k_n$ with $k_1 \geqslant 2$ by

$$\zeta(k_1, \dots, k_n) = \sum_{m_1 > \cdots > m_n > 0} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}}.$$

One of the main goal in the theory of MZV's is to obtain as many identities among them as possible, and to understand the structure of these relations. It is widely conjectured (cf. [2]) that the double shuffle relations, properly extended by introducing the "regularization" of divergent sums and integrals, give all linear (and algebraic) relations among MZV's. In [2], Ihara, Kaneko and Zagier showed that the sum formula, Hoffman's relations, and the derivation relations of the MZV's are subject to this "*extended double shuffle relations*" (EDSR's). Also, they showed the

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EDSR's together with the duality relation implies Ohno's relations. Curiously enough however, it is not yet known whether the duality relation itself, the simplest identity between two "dual" MZV's, is a consequence of the EDSR's.

The purpose of this paper is to show that the relation

$$\sum(\text{MZV's of fixed weight, depth, and height}) = \text{its dual}$$

can be derived from the EDSR's. As a simplest special case, we obtain the duality relation

$$\zeta(m+1,\underbrace{1,\dots,1}_{n-1}) = \zeta(n+1,\underbrace{1,\dots,1}_{m-1})$$

as a consequence of the EDSR's.

2. Algebraic setup and the main theorem (abstract version)

We use the algebraic setup introduced by Hoffman [1] (also adapted in [2]), which encodes the MZV's as monomials in the noncommutative polynomial ring $\mathfrak{h} = \mathbb{Q}\langle x, y\rangle$ over $\mathbb{Q}$ in two indeterminates x and y. Let $\mathfrak{h}^0 = \mathbb{Q} + x\mathfrak{h}y$ be the subring of $\mathfrak{h}$. Define the $\mathbb{Q}$-linear map $Z : \mathfrak{h}^0 \to \mathbb{R}$ by

$$\begin{aligned}
&Z(1) = 1,\\
&Z\big(x^{k_1-1}yx^{k_2-1}y\cdots x^{k_n-1}y\big) = \zeta(k_1,k_2,\dots,k_n).
\end{aligned}$$

The weight $k_1+\dots+k_n$ and the depth n of $\zeta(k_1,k_2,\dots,k_n)$ correspond to the total degree and the degree in y of the monomial $x^{k_1-1}yx^{k_2-1}y\cdots x^{k_n-1}y$, respectively. The length of the monomial viewed as a product of $x^m y^n$ $(m,n \geqslant 1)$ is called height, which is equal to the number of indices k_i grater than 1. To obtain a linear relation of MZV's is to obtain an element in the kernel of Z. The totality of the EDSR's in [2] defines a subspace V_{EDSR} of $\mathfrak{h}^0$ contained in $\ker Z$. The conjecture is $V_{\mathrm{EDSR}} = \ker Z$.

Let τ be the anti-automorphism on $\mathfrak{h} = \mathbb{Q}\langle x, y\rangle$ defined by

$$\tau(x) = y, \qquad \tau(y) = x.$$

The map τ preserves $\mathfrak{h}^0$, and the duality theorem is stated as

$$Z\big(\tau(w_0)\big) = Z(w_0) \quad \big(w_0 \in \mathfrak{h}^0\big).$$

In other words, the duality is the statement that the image of $\mathfrak{h}^0$ under the map $\tau - 1$ is contained in the kernel of Z. Still unknown is $(\tau-1)(\mathfrak{h}^0) \subset V_{\mathrm{EDSR}}$. Our result in this paper is the following.

Main Theorem *(Abstract version). For any fixed weight k, depth n, and height s, we have*

$$(\tau-1)\Bigg(\sum_{\substack{l_1+\dots+l_s=k-n\\ m_1+\dots+m_s=n\\ l_1,\dots,l_s,m_1,\dots,m_s\geqslant 1}} x^{l_1}y^{m_1}x^{l_2}y^{m_2}\cdots x^{l_s}y^{m_s}\Bigg) \in V_{\mathrm{EDSR}}.$$

In the next section we state the theorem in a more precise form and give the proof.

3. The main theorem (precise version) and its proof

For any integer $l \geqslant 0$ we define the map $\theta_l : \mathfrak{h}^0 \to \mathfrak{h}^0$ by

$$\theta_0 = \mathrm{id},$$
$$\theta_l(w_0) = (-1)^l \operatorname{reg}_{Ш}\big(y^l * w_0\big) \quad \big(l \geqslant 1,\ w_0 \in \mathfrak{h}^0\big),$$

and put $\Theta = \sum_{l \geqslant 0} \theta_l$. Here $*$ is the harmonic product (the series shuffle product, see [1]) and the map $\operatorname{reg}_{Ш} : \mathfrak{h}^1 (:= \mathbb{Q} + \mathfrak{h} y) \to \mathfrak{h}^0$ is obtained by taking the constant term of the image of the unique map $Z^{Ш} : \mathfrak{h}^1 \to \mathfrak{h}^0[T]$ which is ш-algebra (ш is the integral shuffle product) homomorphism and sends y to T. By [2], $\theta_l(w_0) \in V_{\mathrm{EDSR}}$ ($l \geqslant 1$, $w_0 \in \mathfrak{h}^0$). We state our precise version of the theorem in terms of θ_l.

Main Theorem *(Precise version). For any integers $n \geqslant s \geqslant 1$, we have*

$$(\tau - 1)\left(\sum_{\substack{m_1+\cdots+m_s=n\\ m_1,\ldots,m_s \geqslant 1}} \prod_{j=1}^{s} \frac{x}{1-x} y^{m_j}\right) = \Theta\big(P_s(n)\big) - \sum_{l=0}^{n-s} \theta_l\big(P_s(n-l)\big)$$
$$= (\Theta - 1)\big(P_s(n)\big) - \sum_{l=1}^{n-s} \theta_l\big(P_s(n-l)\big),$$

where

$$P_s(m) = \sum_{\substack{i_1+\cdots+i_s=m\\ i_1,\ldots,i_s \geqslant 1}} \prod_{j=1}^{s} \left\{\left(x - \frac{x}{1-x} y\right)^{i_j - 1} \frac{x}{1-x} y\right\}.$$

Here, we have naturally extended Θ, θ_l to $\hat{\mathfrak{h}}^0$, the completion of $\mathfrak{h}^0$ with respect to degree, and we always understand the noncommutative product $\prod_{j=1}^{s} f_j(x,y)$ to be $f_1(x,y) \times f_2(x,y) \cdots f_s(x,y)$.

By comparing the homogeneous components of both sides of the theorem, we obtain an expression of the duality of the sum of fixed weight, depth and height as a linear combination of $\theta_l(w_0)$.

Remark. The quantity in the theorem is exactly equal to that in the main theorem of Ohno–Zagier [3]. They proved that this quantity is a polynomial of Riemann zeta values.

Example. Case $n = 3$, $s = 2$:

$$\begin{aligned}
&x^2\frac{y}{1-y}x\frac{y}{1-y}+x\frac{y}{1-y}x^2\frac{y}{1-y}-\frac{x}{1-x}y\frac{x}{1-x}y^2-\frac{x}{1-x}y^2\frac{x}{1-x}y\\
&\quad=(\Theta-1)\left(\left(x-\frac{x}{1-x}y\right)\frac{x}{1-x}y\frac{x}{1-x}y+\frac{x}{1-x}y\left(x-\frac{x}{1-x}y\right)\frac{x}{1-x}y\right)\\
&\qquad-\theta_1\left(\frac{x}{1-x}y\frac{x}{1-x}y\right),
\end{aligned}$$

$$\begin{aligned}
&x^2yxy+xyx^2y-xyxy^2-xy^2xy=-\theta_1(xyxy)\\
&\quad\longrightarrow\quad \zeta(3,2)+\zeta(2,3)-\zeta(2,2,1)-\zeta(2,1,2)=0,
\end{aligned}$$

$$\begin{aligned}
&x^2y^2xy+x^2yxy^2+xy^2x^2y+xyx^2y^2-x^2yxy^2-xyx^2y^2-x^2y^2xy-xy^2x^2y\\
&\quad=\theta_1\big(x^2yxy\big)+\theta_1\big(xyx^2y\big)-\theta_1\big(x^2yxy\big)-\theta\big(xyx^2y\big)=0\\
&\quad\longrightarrow\quad \text{self dual},
\end{aligned}$$

$$\begin{aligned}
&x^2y^3xy+x^2y^2xy^2+x^2yxy^3+xy^3x^2y+xy^2x^2y^2+xyx^2y^3\\
&\qquad-x^3yxy^2-x^2yx^2y^2-xyx^3y^2-x^3y^2xy-x^2y^2x^2y-xy^2x^3y\\
&\quad=\theta_1\big(x^3yxy\big)+\theta_1\big(x^2yx^2y\big)+\theta_2\big(x^2yxy\big)-2\theta_1(xyxyxy)\\
&\qquad+\theta_1\big(x^2yx^2y\big)+\theta_1\big(xyx^3y\big)+\theta_2\big(xyx^2y\big)-\theta_1\big(x^3yxy\big)-\theta_1\big(x^2yx^2y\big)-\theta_1\big(xyx^3y\big)\\
&\quad=\theta_1\big(x^2yx^2y\big)-2\theta_1(xyxyxy)+\theta_2\big(x^2yxy\big)+\theta_2\big(xyx^2y\big)\\
&\quad\longrightarrow\quad \zeta(3,1,1,2)+\zeta(3,1,2,1)+\zeta(3,2,1,1)+\zeta(2,1,1,3)+\zeta(2,1,3,1)\\
&\qquad\quad+\zeta(2,3,1,1)-\zeta(4,2,1)-\zeta(3,3,1)-\zeta(2,4,1)-\zeta(4,1,2)-\zeta(3,1,3)\\
&\qquad\quad-\zeta(2,1,4)=0,\\
&\quad\vdots
\end{aligned}$$

Proof. Let $\mathfrak{h}[\![u]\!]$ be the formal power series ring over $\mathfrak{h}$ with an indeterminate u, and Δ_u the automorphism of $\mathfrak{h}[\![u]\!]$ whose images of the generators are

$$\Delta_u(u)=u,\qquad \Delta_u(x)=x\frac{1}{1-yu},\qquad \Delta_u(y)=(1-xu-yu)\frac{y}{1-yu}.$$

The images of the generators of the inverse map Δ_u^{-1} are then given by

$$\Delta_u^{-1}(u)=u,\qquad \Delta_u^{-1}(x)=\frac{x}{1-xu}(1-xu-yu),\qquad \Delta_u^{-1}(y)=\frac{1}{1-xu}y. \tag{1}$$

From [2, Corollary of Proposition 7], changing u into $-u$, we have

$$\Delta_u(w_0)=\mathrm{reg}_{\text{Ш}}\left(\frac{1}{1+yu}*w_0\right)=\left(\sum_{l=0}^{\infty}\theta_l u^l\right)(w_0)\quad\big(w_0\in\mathfrak{h}^0\big). \tag{2}$$

Putting $u = 1$, we see that the map Δ_1 (which naturally extends to the automorphism of $\hat{\mathfrak{h}}$, the completion of $\mathfrak{h}$) coincides with Θ on $\hat{\mathfrak{h}}^0$. Hence, we may extend Θ to the automorphism of $\hat{\mathfrak{h}}$ via the identification with Δ_1. Therefore, to prove the main theorem, we have to show

$$\left(\sum_{\substack{m_1+\cdots+m_s=n\\ m_1,\ldots,m_s\geqslant 1}}\prod_{j=1}^{s}\frac{x}{1-x}y^{m_j}\right)=\sum_{l=0}^{n-s}\theta_l\bigl(P_s(n-l)\bigr).$$

We compute the generating functions of the both sides of the above identity by multiplying u^{n-s} and taking the sum $\sum_{n\geqslant s}^{\infty}$. The left-hand side then becomes

$$\begin{aligned}
&\sum_{n\geqslant s}\left(\sum_{\substack{k_1+\cdots+k_s=n\\ k_1,\ldots,k_s\geqslant 1}}\prod_{j=1}^{s}\frac{x}{1-x}y^{k_j}\right)u^{n-s}\\
&\quad=\sum_{n\geqslant 0}\left(\sum_{\substack{k_1+\cdots+k_s=n+s\\ k_1,\ldots,k_s\geqslant 1}}\prod_{j=1}^{s}\frac{x}{1-x}y^{k_j}\right)u^{n}\\
&\quad=\sum_{n\geqslant 0}\left(\sum_{\substack{k_1+\cdots+k_s=n\\ k_1,\ldots,k_s\geqslant 0}}\prod_{j=1}^{s}\frac{x}{1-x}y^{k_j+1}\right)u^{n}=\sum_{k_1,\ldots,k_s\geqslant 0}\prod_{j=1}^{s}\frac{x}{1-x}y(yu)^{k_j}\\
&\quad=\prod_{j=1}^{s}\frac{x}{1-x}\frac{y}{1-yu}=\left(\frac{x}{1-x}\frac{y}{1-yu}\right)^{s}.
\end{aligned}$$

By (2), the right-hand side becomes

$$\begin{aligned}
&\sum_{n\geqslant s}\sum_{l=0}^{n-s}\theta_l\left(\sum_{\substack{i_1+\cdots+i_s=n-l\\ i_1,\ldots,i_s\geqslant 1}}\prod_{j=1}^{s}\left\{\left(x-\frac{x}{1-x}y\right)^{i_j-1}\frac{x}{1-x}y\right\}\right)u^{n-s}\\
&\quad=\sum_{n\geqslant 0}\sum_{l=0}^{n}\theta_l\left(\sum_{\substack{i_1+\cdots+i_s=n+s-l\\ i_1,\ldots,i_s\geqslant 1}}\prod_{j=1}^{s}\left\{\left(x-\frac{x}{1-x}y\right)^{i_j-1}\frac{x}{1-x}y\right\}\right)u^{n}\\
&\quad=\sum_{l\geqslant 0}\theta_l\left(\sum_{n\geqslant l}\sum_{\substack{i_1+\cdots+i_s=n-l\\ i_1,\ldots,i_s\geqslant 0}}\prod_{j=1}^{s}\left\{\left(x-\frac{x}{1-x}y\right)^{i_j}\frac{x}{1-x}y\right\}\right)u^{n}\\
&\quad=\sum_{l\geqslant 0}\theta_l\left(\sum_{n\geqslant 0}\sum_{\substack{i_1+\cdots+i_s=n\\ i_1,\ldots,i_s\geqslant 0}}\prod_{j=1}^{s}\left\{\left(x-\frac{x}{1-x}y\right)^{i_j}\frac{x}{1-x}y\right\}\right)u^{n+l}\\
&\quad=\left(\sum_{l\geqslant 0}\theta_l u^l\right)\left(\sum_{i_1,\ldots,i_s\geqslant 0}\prod_{j=1}^{s}\left[\left\{\left(x-\frac{x}{1-x}y\right)u\right\}^{i_j}\frac{x}{1-x}y\right]\right)
\end{aligned}$$

$$= \varDelta_u\left(\prod_{j=1}^{s}\left\{\frac{1}{1-(x-\frac{x}{1-x}y)u}\,\frac{x}{1-x}y\right\}\right) = \varDelta_u\left(\frac{1}{1-x-x(1-x-y)u}xy\right)^s.$$

Using (1), it is straightforward to show

$$\varDelta_u^{-1}\left(\frac{x}{1-x}\,\frac{y}{1-yu}\right) = \frac{1}{1-x-x(1-x-y)u}xy.$$

This shows that the generating functions of the both sides are equal, and the theorem is proved. $\square$

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